Symplectic integration methods for multidimensional disordered nonlinear lattices

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Outline

- **Symplectic Integrators: Application to disordered nonlinear lattices**
  - ✓ **Models**
    - **The quartic Klein-Gordon (KG) disordered lattice**
    - **The disordered discrete nonlinear Schrödinger equation (DNLS)**
  - ✓ **Symplectic integration of KG and DNLS models**
  - ✓ **Numerical results: different dynamical behaviors**

- **Symplectic integration of variational equations: The Tangent Map (TM) method**
  - ✓ **Chaos indicators**
    - **Lyapunov exponents**
    - **Generalized Alignment Index (GALI)**
  - ✓ **Chaotic behavior of the KG model**

- **High order three part split symplectic integrators for the DNLS model**

- **Outlook**

Autonomous Hamiltonian systems

Consider an N degree of freedom autonomous Hamiltonian system having a Hamiltonian function of the form:

$$H(\underbrace{q_1,q_2,\ldots,q_N}_{\text{positions}},\ \underbrace{p_1,p_2,\ldots,p_N}_{\text{momenta}})$$

The time evolution of an orbit (trajectory) with initial condition

$$P(0)=(q_1(0),\ q_2(0),\ldots,q_N(0),\ p_1(0),\ p_2(0),\ldots,p_N(0))$$

is governed by the Hamilton's equations of motion

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\ ,\quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

Symplectic Integration schemes

Formally the solution of the Hamilton equations of motion can be written as:

$$\frac{d\vec{X}}{dt} = \left\{H, \vec{X}\right\} = L_H \vec{X} \Rightarrow \vec{X}(t) = \sum_{n\geq 0} \frac{t^n}{n!} L_H^n \vec{X} = e^{tL_H} \vec{X}$$

where $\vec{X}$ is the full coordinate vector and L_H the Poisson operator:

$$L_H f = \sum_{j=1}^{N} \left\{ \frac{\partial H}{\partial p_j} \frac{\partial f}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial f}{\partial p_j} \right\}$$

If the Hamiltonian H can be split into two integrable parts as H=A+B, a symplectic scheme for integrating the equations of motion from time t to time t+τ consists of approximating the operator $e^{\tau L_H}$ by

$$e^{\tau L_H} = e^{\tau(L_A + L_B)} = \prod_{i=1}^{j} e^{c_i \tau L_A} e^{d_i \tau L_B} + O(\tau^{n+1})$$

for appropriate values of constants c_i, d_i. This is an integrator of order n.

So the dynamics over an integration time step τ is described by a series of successive acts of Hamiltonians A and B.

Symplectic Integrator SABA$_2$C

The operator $e^{\tau L_H}$ can be approximated by the symplectic integrator [Laskar & Robutel, Cel. Mech. Dyn. Astr. (2001)]:

$$SABA_2 = e^{c_1 \tau L_A}\ e^{d_1 \tau L_B}\ e^{c_2 \tau L_A}\ e^{d_1 \tau L_B}\ e^{c_1 \tau L_A}$$

with $c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6},\ c_2 = \frac{\sqrt{3}}{3},\ d_1 = \frac{1}{2}.$

The integrator has only small positive steps and its error is of order 2.

In the case where *A* is quadratic in the momenta and *B* depends only on the positions the method can be improved by introducing a corrector *C*, having a small negative step:

$$C = e^{-\tau^3 \frac{c}{2} L_{\{\{A,B\},B\}}}$$

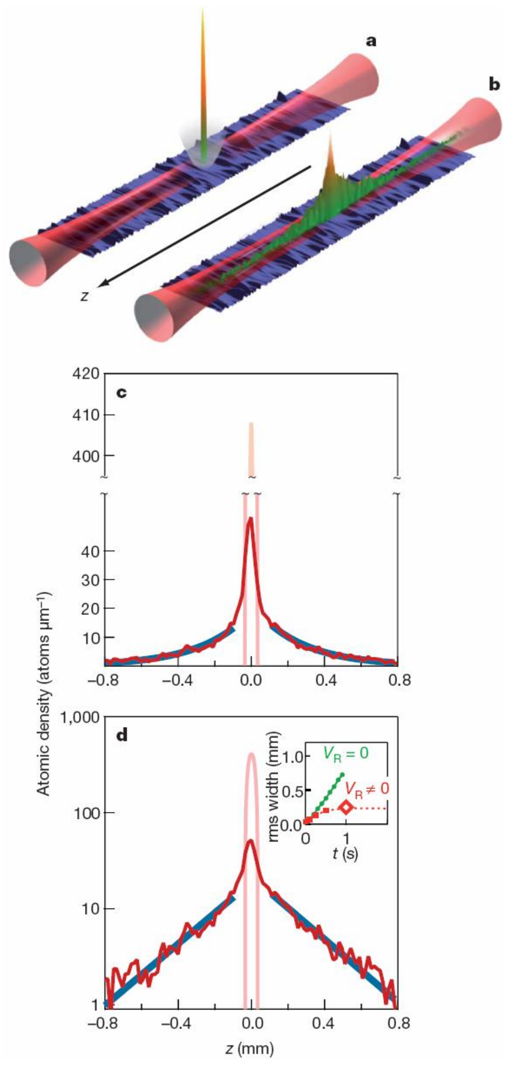
with $c = \frac{2 - \sqrt{3}}{24}.$

Thus the full integrator scheme becomes: $SABAC_2 = C\ (SABA_2)\ C$ and its error is of order 4.

Interplay of disorder and nonlinearity

Waves in disordered media – Anderson localization

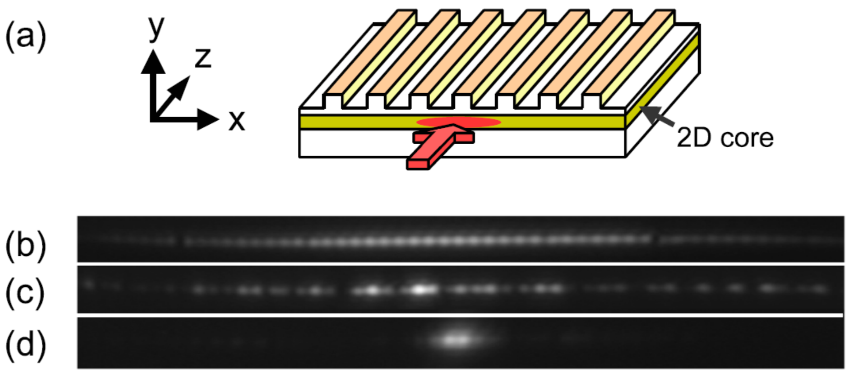
[Anderson, Phys. Rev. (1958)]. Experiments on BEC [Billy et al., Nature (2008)]



Waves in nonlinear disordered media – localization or delocalization?

Theoretical and/or numerical studies [Shepelyansky, PRL, (1993) – Molina, Phys. Rev. B (1998) - Pikovsky & Shepelyansky, PRL, (2008) - Kopidakis et al., PRL, (2008)]

Experiments: propagation of light in disordered 1d waveguide lattices [Lahini et al., PRL, (2008)]



The Klein – Gordon (KG) model

$$H_K = \sum_{l=1}^{N} \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} \left(u_{l+1} - u_l \right)^2$$

with fixed boundary conditions $u_0=p_0=u_{N+1}=p_{N+1}=0$. Typically N=1000.

Parameters: W and the total energy E. $\tilde{\varepsilon}_l$ chosen uniformly from $\left[\frac{1}{2}, \frac{3}{2}\right]$.

The discrete nonlinear Schrödinger (DNLS) equation

We also consider the system:

$$H_D = \sum_{l=1}^{N} \varepsilon_l |\psi_l|^2 + \frac{\beta}{2} |\psi_l|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l \right)$$

where ε_l chosen uniformly from $\left[-\frac{W}{2}, \frac{W}{2}\right]$ and β is the nonlinear parameter.

Conserved quantities: The energy and the norm of the wave packet.

Distribution characterization

We consider normalized energy distributions in normal mode (NM) space

$$z_\nu \equiv \frac{E_\nu}{\sum_m E_m} \quad \text{with} \quad E_\nu = \frac{1}{2}\left(\dot{A}_\nu^2 + \omega_\nu^2 A_\nu^2\right)$$

, where A_ν is the amplitude of the νth NM.

Second moment: $$m_2 = \sum_{\nu=1}^{N} (\nu - \bar{\nu})^2 z_\nu \quad \text{with} \quad \bar{\nu} = \sum_{\nu=1}^{N} \nu z_\nu$$

Participation number: $$P = \frac{1}{\sum_{\nu=1}^{N} z_\nu^2}$$

measures the number of stronger excited modes in z_ν. Single mode $P=1$, Equipartition of energy $P=N$.

The KG model

We apply the SABAC$_2$ integrator scheme to the KG Hamiltonian by using the splitting:

$$H_K = \sum_{l=1}^{N} \Bigg(\underbrace{\frac{p_l^2}{2}}_{A} + \underbrace{\frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W}\left(u_{l+1} - u_l\right)^2}_{B} \Bigg)$$

$$e^{\tau L_A}: \begin{cases} u_l' = p_l \tau + u_l \\ p_l' = p_l, \end{cases}$$

$$e^{\tau L_B}: \begin{cases} u_l' = u_l \\ p_l' = \left[-u_l(\tilde{\epsilon}_l + u_l^2) + \frac{1}{W}(u_{l-1} + u_{l+1} - 2u_l) \right] \tau + p_l, \end{cases}$$

with a corrector term which corresponds to the Hamiltonian function:

$$\mathrm{C} = \left\{ \left\{ A, B \right\}, B \right\} = \sum_{l=1}^{N} \left[u_l(\tilde{\varepsilon}_l + u_l^2) - \frac{1}{W}(u_{l-1} + u_{l+1} - 2u_l) \right]^2 .$$

The DNLS model

A 2nd order SABA Symplectic Integrator with 5 steps, combined with approximate solution for the B part (Fourier Transform): SIFT_2

$$H_D = \sum_l \varepsilon_l |\psi_l|^2 + \frac{\beta}{2}|\psi_l|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l\right), \qquad \psi_l = \frac{1}{\sqrt{2}}\left(q_l + ip_l\right)$$

$$H_D = \sum_l \Bigg(\underbrace{\frac{\varepsilon_l}{2}\left(q_l^2 + p_l^2\right) + \frac{\beta}{8}\left(q_l^2 + p_l^2\right)^2}_{A} \underbrace{- q_n q_{n+1} - p_n p_{n+1}}_{B} \Bigg)$$

$$e^{\tau L_A}: \begin{cases} q_l' = q_l \cos(\alpha_l \tau) + p_l \sin(\alpha_l \tau), \\ p_l' = p_l \cos(\alpha_l \tau) - q_l \sin(\alpha_l \tau), \end{cases}$$

$$\alpha_l = \epsilon_l + \beta(q_l^2 + p_l^2)/2$$

$$e^{\tau L_B}: \begin{cases} \varphi_q = \sum_{m=1}^{N} \psi_m e^{2\pi i q(m-1)/N} \\ \varphi_q' = \varphi_q e^{2i\cos(2\pi(q-1)/N)\tau} \\ \psi_l' = \frac{1}{N}\sum_{q=1}^{N} \varphi_q' e^{-2\pi i l(q-1)/N} \end{cases}$$

The DNLS model

Symplectic Integrators produced by Successive Splits (SS)

$$H_D = \sum_l \left(\underbrace{\frac{\varepsilon_l}{2}\left(q_l^2 + p_l^2\right) + \frac{\beta}{8}\left(q_l^2 + p_l^2\right)^2}_{A} \underbrace{- q_n q_{n+1} - p_n p_{n+1}}_{B} \right)$$

A:

$$\begin{cases} q_l' = q_l \cos(\alpha_l \tau) + p_l \sin(\alpha_l \tau), \\ p_l' = p_l \cos(\alpha_l \tau) - q_l \sin(\alpha_l \tau), \end{cases}$$

B_1:

$$\begin{cases} q_l' = q_l, \\ p_l' = p_l + (q_{l-1} + q_{l+1})\tau \end{cases}$$

B_2:

$$\begin{cases} p_l' = p_l, \\ q_l' = q_l - (p_{l-1} + p_{l+1})\tau \end{cases}$$

Using the $SABA_2$ integrator we get a 2nd order integrator with 13 steps, $SS(SABA_2)_2$:

$$SS(SABA_2)_2 = e^{\left[\frac{(3-\sqrt{3})}{6}\tau\right] L_A} \, e^{\frac{\tau}{2} L_B} \, e^{\frac{\sqrt{3}\tau}{3} L_A} \, e^{\frac{\tau}{2} L_B} \, e^{\left[\frac{(3-\sqrt{3})}{6}\tau\right] L_A}$$

$\tau' = \tau / 2$

$$e^{\frac{\tau}{2} L_B} = e^{\left[\frac{(3-\sqrt{3})}{6}\tau'\right] L_{B_1}} \, e^{\frac{\tau'}{2} L_{B_2}} \, e^{\frac{\sqrt{3}\tau'}{3} L_{B_1}} \, e^{\frac{\tau'}{2} L_{B_2}} \, e^{\left[\frac{(3-\sqrt{3})}{6}\tau'\right] L_{B_1}}$$

Different Dynamical Regimes

Three expected evolution regimes [Flach, Chem. Phys (2010) - Ch.S. & Flach, PRE (2010) - Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE (2011)]

Δ: width of the frequency spectrum, d: average spacing of interacting modes, δ : nonlinear frequency shift.

Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$

Frequency shift is less than the average spacing of interacting modes. NMs are weakly interacting with each other. [Molina, PRB (1998) – Pikovsky, & Shepelyansky, PRL (2008)].

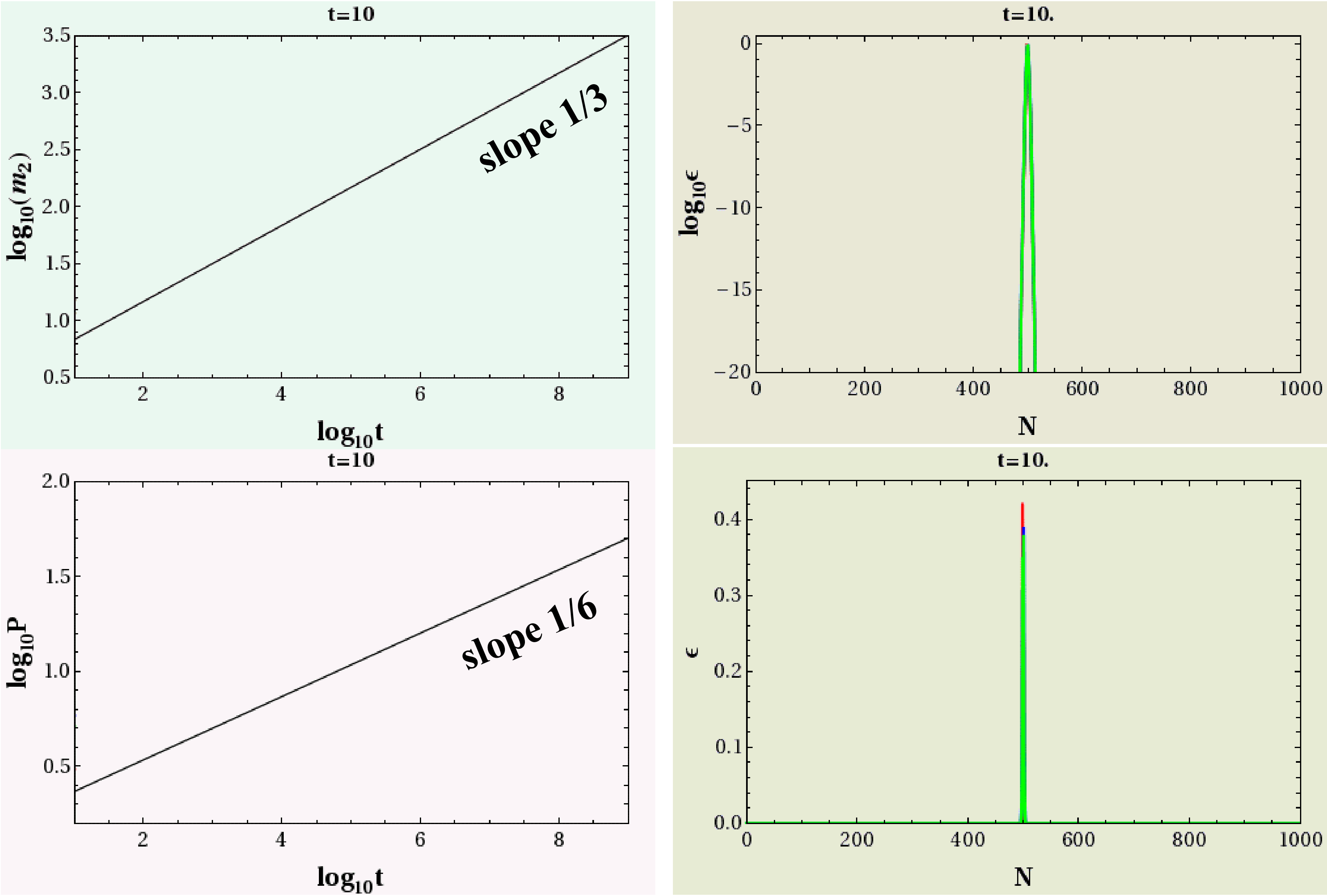
Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2} \longrightarrow m_2 \sim t^{1/3}$

Almost all NMs in the packet are resonantly interacting. Wave packets initially spread faster and eventually enter the weak chaos regime.

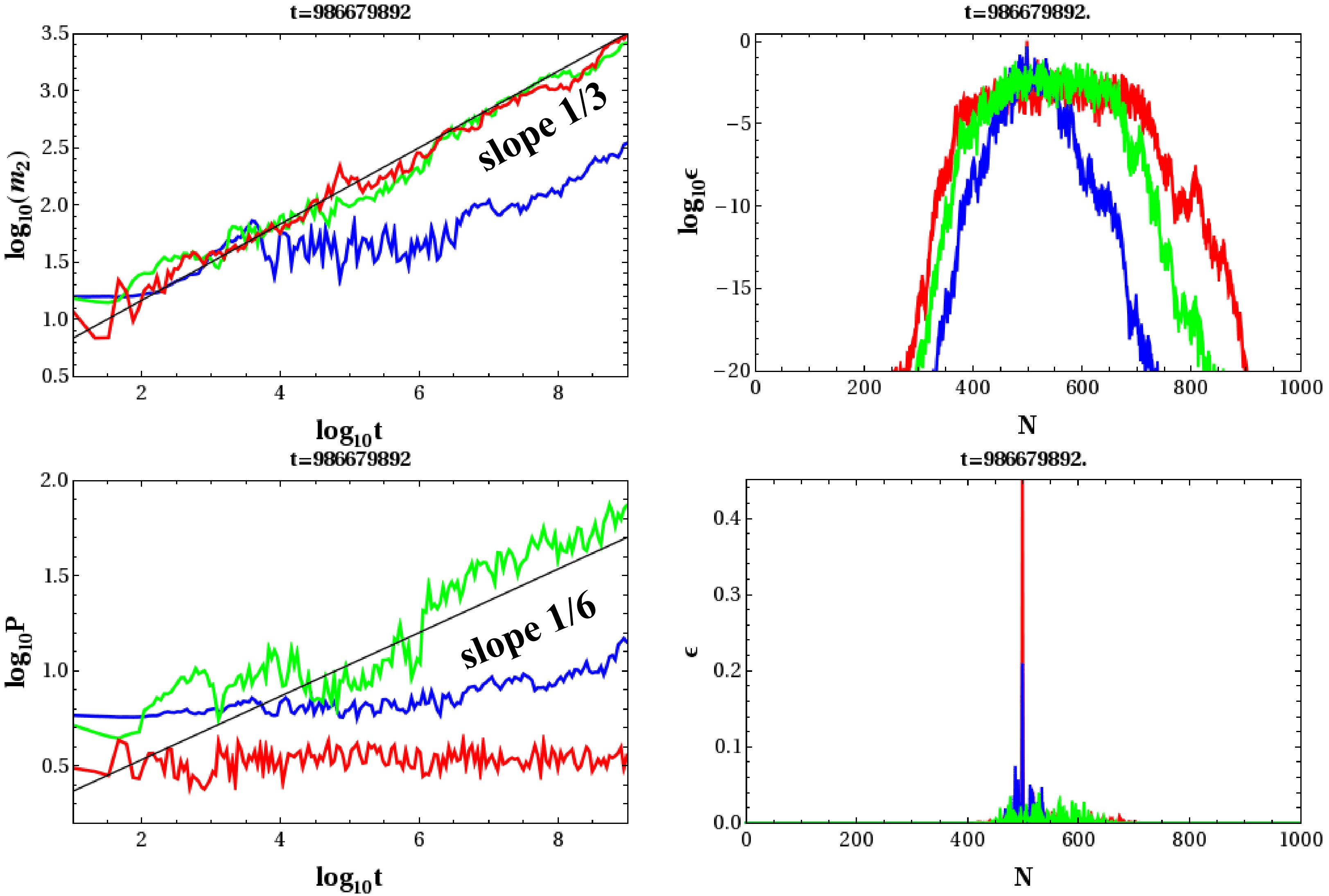
Selftrapping Regime: $\delta>\Delta$

Frequency shift exceeds the spectrum width. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, leading to selftrapping, while a small part of the wave packet subdiffuses [Kopidakis et al., PRL (2008)].

Different spreading regimes

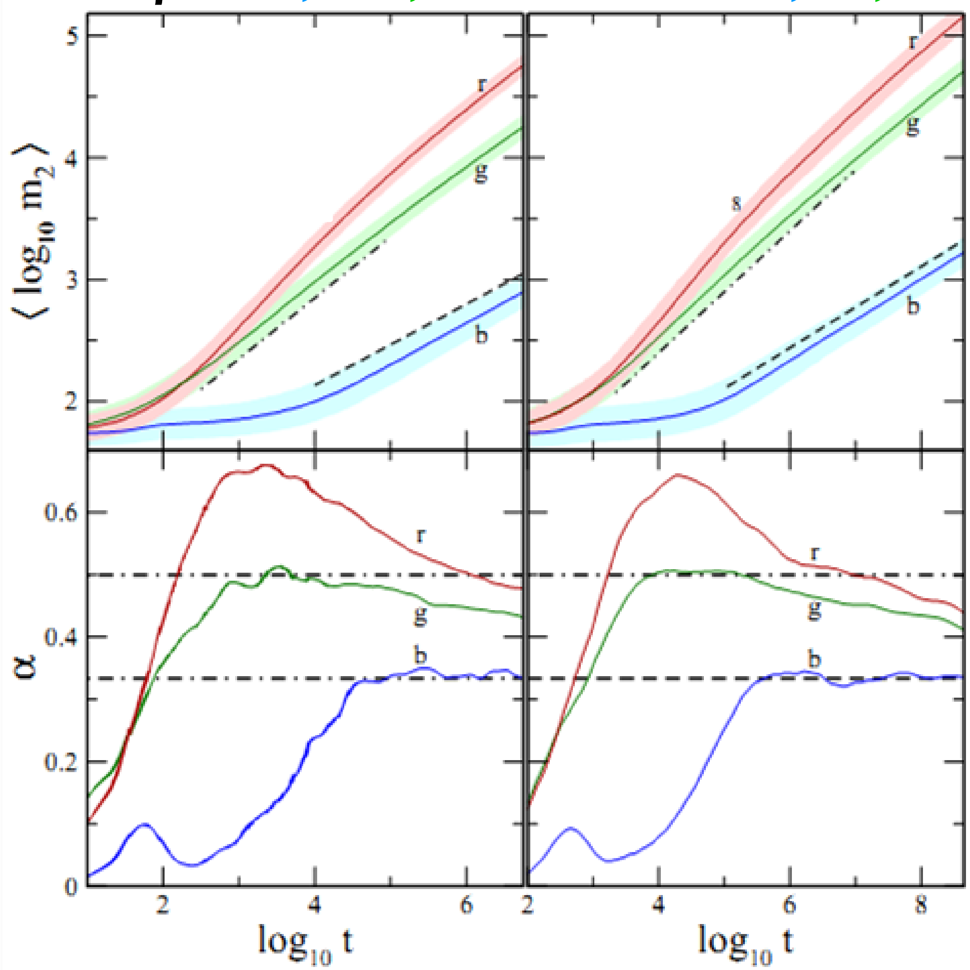


Different spreading regimes



Crossover from strong to weak chaos

DNLS β= 0.04, 0.72, 3.6 KG E= 0.01, 0.2, 0.75



α=1/2

α=1/3

W=4

Average over 1000 realizations!

$$\alpha(\log t) = \frac{d \langle \log m_2 \rangle}{d \log t}$$

Symplectic integration of variational equations

Autonomous Hamiltonian systems

We study N degree of freedom autonomous Hamiltonian systems of the form:

$$H(\vec{q},\vec{p}) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + V(\vec{q})$$

As an example, we consider the Hénon-Heiles system:

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

Hamilton equations of motion:

$$\begin{cases} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= y^2 - x^2 - y \end{cases}$$

Variational equations:

$$\begin{cases} \dot{\delta x} &= \delta p_x \\ \dot{\delta y} &= \delta p_y \\ \dot{\delta p}_x &= -(1+2y)\delta x - 2x\delta y \\ \dot{\delta p}_y &= -2x\delta x + (-1+2y)\delta y \end{cases}$$

Chaos detection methods

The maximum Lyapunov exponent of a given orbit characterizes the mean exponential rate of divergence of trajectories surrounding this orbit.

$$\mathrm{mLCE} = \lambda_1 = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\vec{w}(t)\|}{\|\vec{w}(0)\|}$$

$\lambda_1 = 0 \rightarrow$ **Regular motion** ($\propto t^{-1}$)

$\lambda_1 \neq 0 \rightarrow$ **Chaotic motion**

Following the evolution of k deviation vectors with $2 \leq k \leq 2N$, we define (Ch.S. et al., 2007) the Generalized Alignment Index (GALI) of order k :

$$\mathrm{GALI}_k(t) = \|\hat{w}_1(t) \wedge \hat{w}_2(t) \wedge \ldots \wedge \hat{w}_k(t)\|$$

Chaotic motion:

$$\mathrm{GALI}_k(t) \propto e^{-[(\lambda_1-\lambda_2)+(\lambda_1-\lambda_3)+\ldots+(\lambda_1-\lambda_k)]t}$$

Regular motion on an s-dimensional torus with $s \leq N$:

$$\mathrm{GALI}_k(t) \propto \begin{cases} \text{constant} & \text{if} \quad 2 \leq k \leq s \\ \dfrac{1}{t^{k-s}} & \text{if} \quad s < k \leq 2N-s \\ \dfrac{1}{t^{2(k-N)}} & \text{if} \quad 2N-s < k \leq 2N \end{cases}$$

Tangent Map (TM) Method

Use symplectic integration schemes for the whole set of equations [Ch.S. & Gerlach, PRE (2010) - Gerlach & Ch.S., Discr. Cont. Dyn. Sys. Supp. (2011) – Gerlach et al., Int. J. Bif. Chaos (2012)]

We apply the SABAC$_2$ integrator scheme to the Hénon-Heiles system by using the splitting:

$$A = \frac{1}{2}(p_x^2 + p_y^2), \qquad B = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3,$$

with a corrector term which corresponds to the Hamiltonian function:

$$C = \{\{A, B\}, B\} = (x + 2xy)^2 + \left(x^2 - y^2 + y\right)^2$$

We approximate the dynamics by the act of Hamiltonians A, B and C, which correspond to the symplectic maps:

$$e^{\tau L_A} : \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ p_x' &= p_x \\ p_y' &= p_y \end{cases},$$

$$e^{\tau L_B} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1 + 2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \end{cases},$$

$$e^{\tau L_C} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1 + 2x^2 + 6y + 2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2 y)\tau \end{cases}.$$

Tangent Map (TM) Method

Any symplectic integration scheme used for solving the Hamilton equations of motion, which involves the act of Hamiltonians A, B and C, can be extended in order to integrate simultaneously the variational equations.

$$e^{\tau L_A} : \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ p_x' &= p_x \\ p_y' &= p_y \end{cases} \quad \rightarrow \quad e^{\tau L_{AV}} : \begin{cases} x' &= x + p_x \tau \\ y' &= y + p_y \tau \\ px' &= p_x \\ py' &= p_y \\ \delta x' &= \delta x + \delta p_x \tau \\ \delta y' &= \delta y + \delta p_y \tau \\ \delta p_x' &= \delta p_x \\ \delta p_y' &= \delta p_y \end{cases}$$

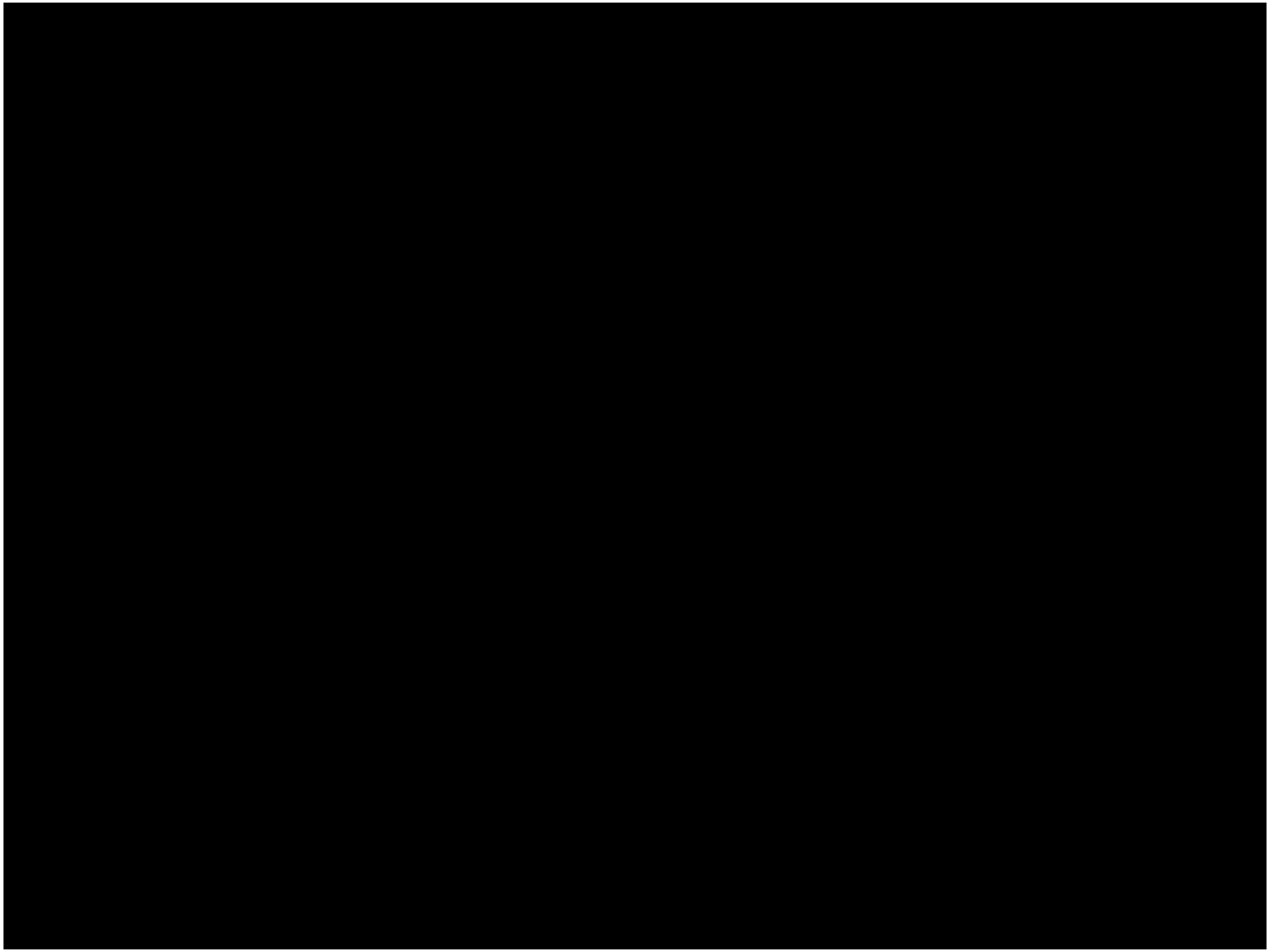
$$e^{\tau L_B} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1+2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \end{cases} \quad \rightarrow \quad e^{\tau L_{BV}} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - x(1+2y)\tau \\ p_y' &= p_y + (y^2 - x^2 - y)\tau \\ \delta x' &= \delta x \\ \delta y' &= \delta y \\ \delta p_x' &= \delta p_x - \left[(1+2y)\delta x + 2x\delta y\right]\tau \\ \delta p_y' &= \delta p_y + \left[-2x\delta x + (-1+2y)\delta y\right]\tau \end{cases}$$

$$e^{\tau L_C} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1 + 2x^2 + 6y + 2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2 y)\tau \end{cases} \quad \rightarrow \quad e^{\tau L_{CV}} : \begin{cases} x' &= x \\ y' &= y \\ p_x' &= p_x - 2x(1 + 2x^2 + 6y + 2y^2)\tau \\ p_y' &= p_y - 2(y - 3y^2 + 2y^3 + 3x^2 + 2x^2 y)\tau \\ \delta x' &= \delta x \\ \delta y' &= \delta y \\ \delta p_x' &= \delta p_x - 2\left[(1 + 6x^2 + 2y^2 + 6y)\delta x + \right. \\ & \quad \left. + 2x(3+2y)\delta y\right]\tau \\ \delta p_y' &= \delta p_y - 2\left[2x(3+2y)\delta x + \right. \\ & \quad \left. + (1 + 2x^2 + 6y^2 - 6y)\delta y\right]\tau \end{cases}$$

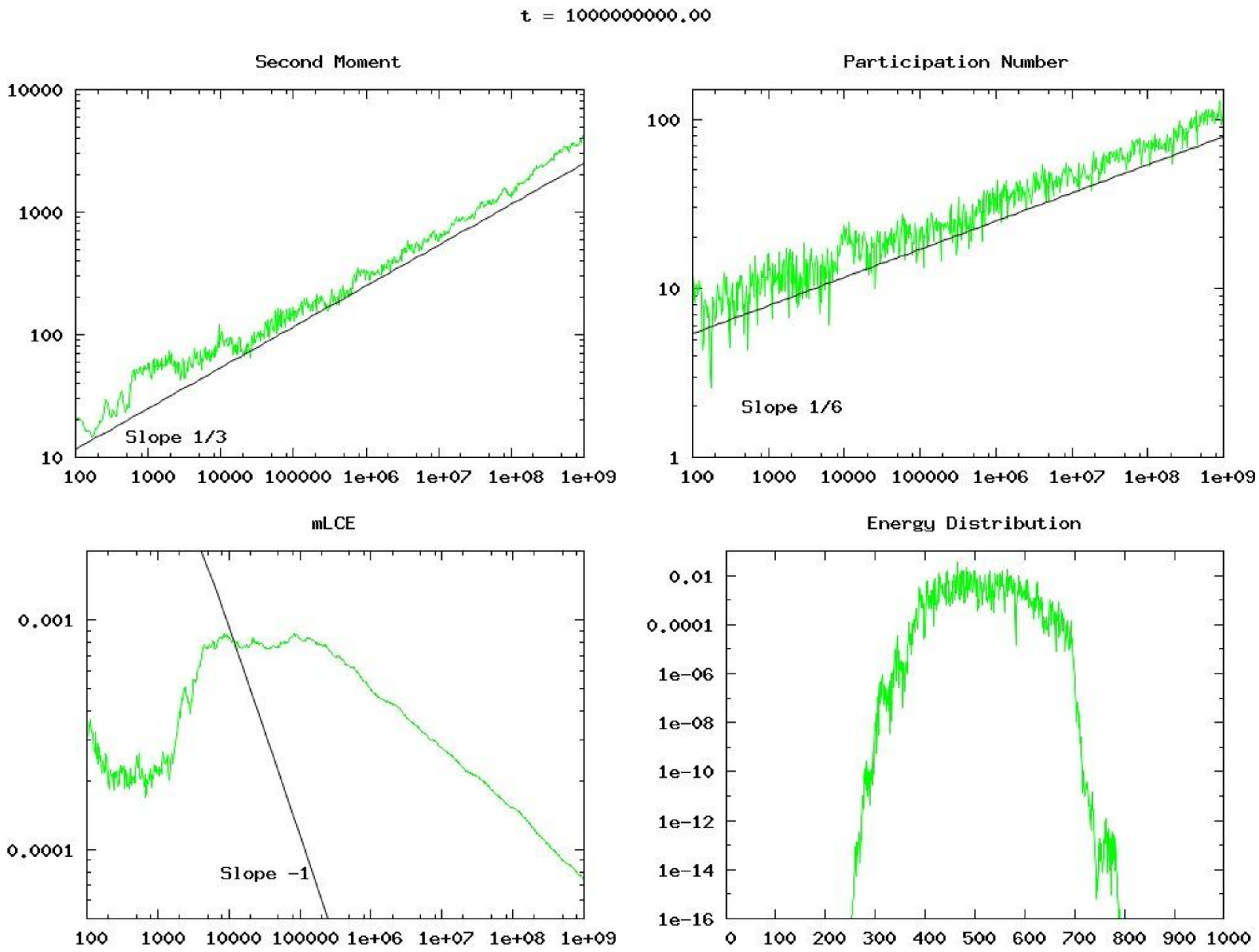
Chaotic behavior of the KG model

Ch.S., I. Gkolias & S. Flach, 2012 (in preparation)

KG: Weak Chaos (E=0.4)



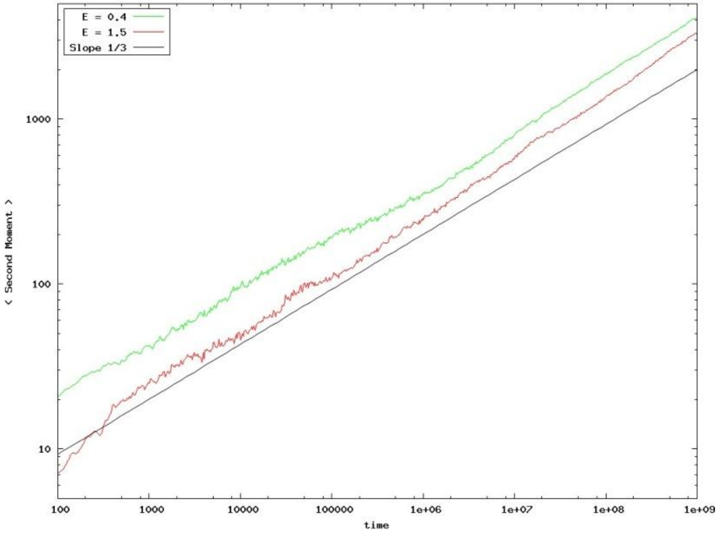
KG: Weak Chaos (E=0.4)

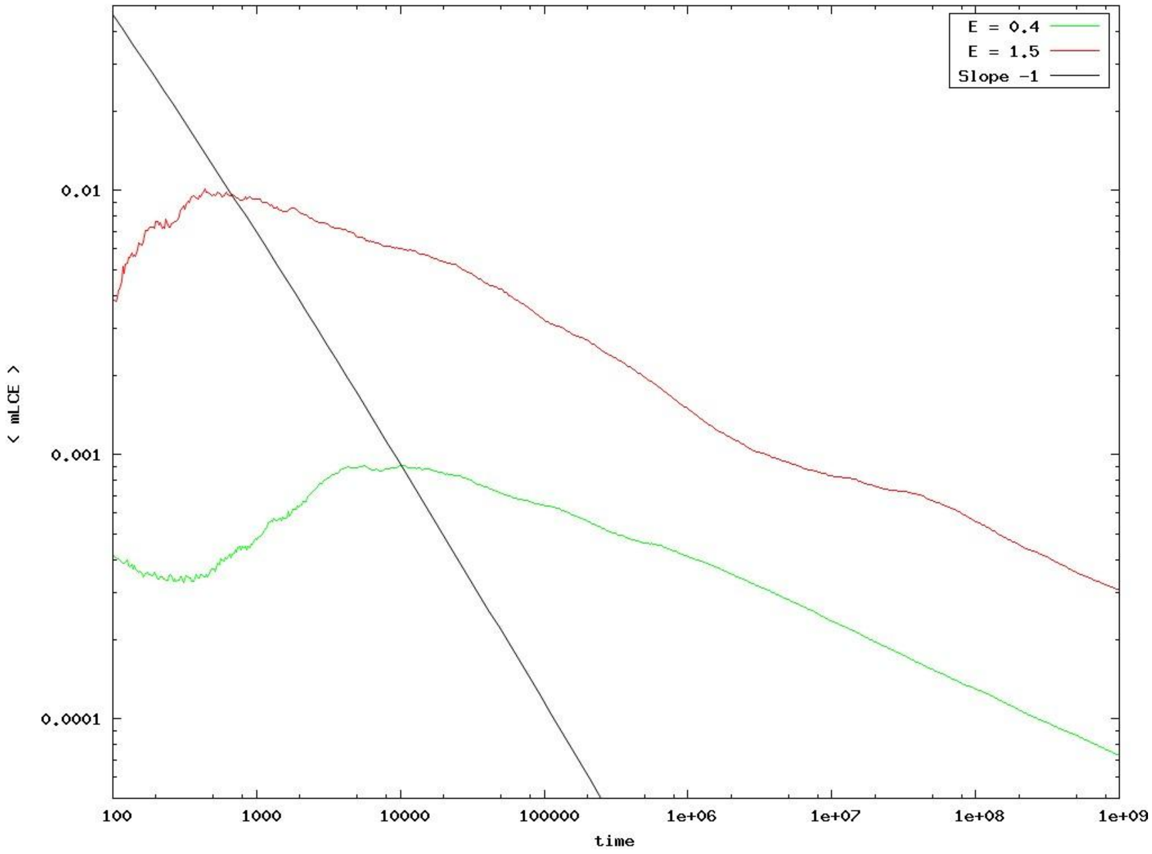


Single site excitations: Different spreading regimes

KG W = 4, E = 0.4, 1.5

Average over 20 realizations

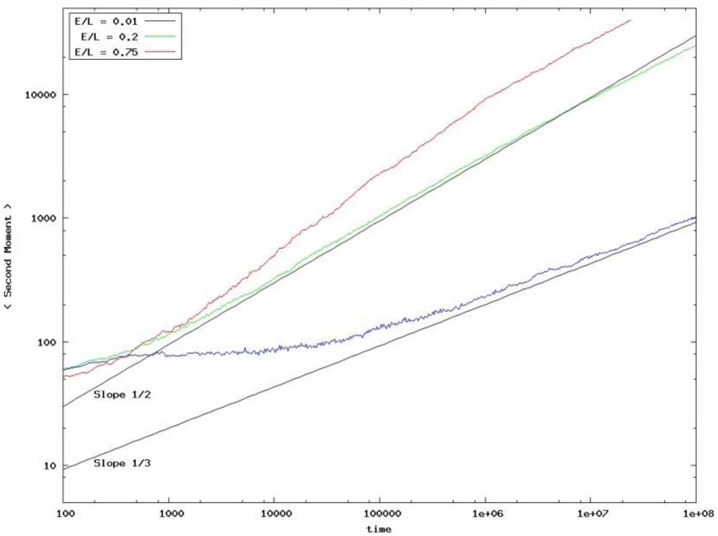


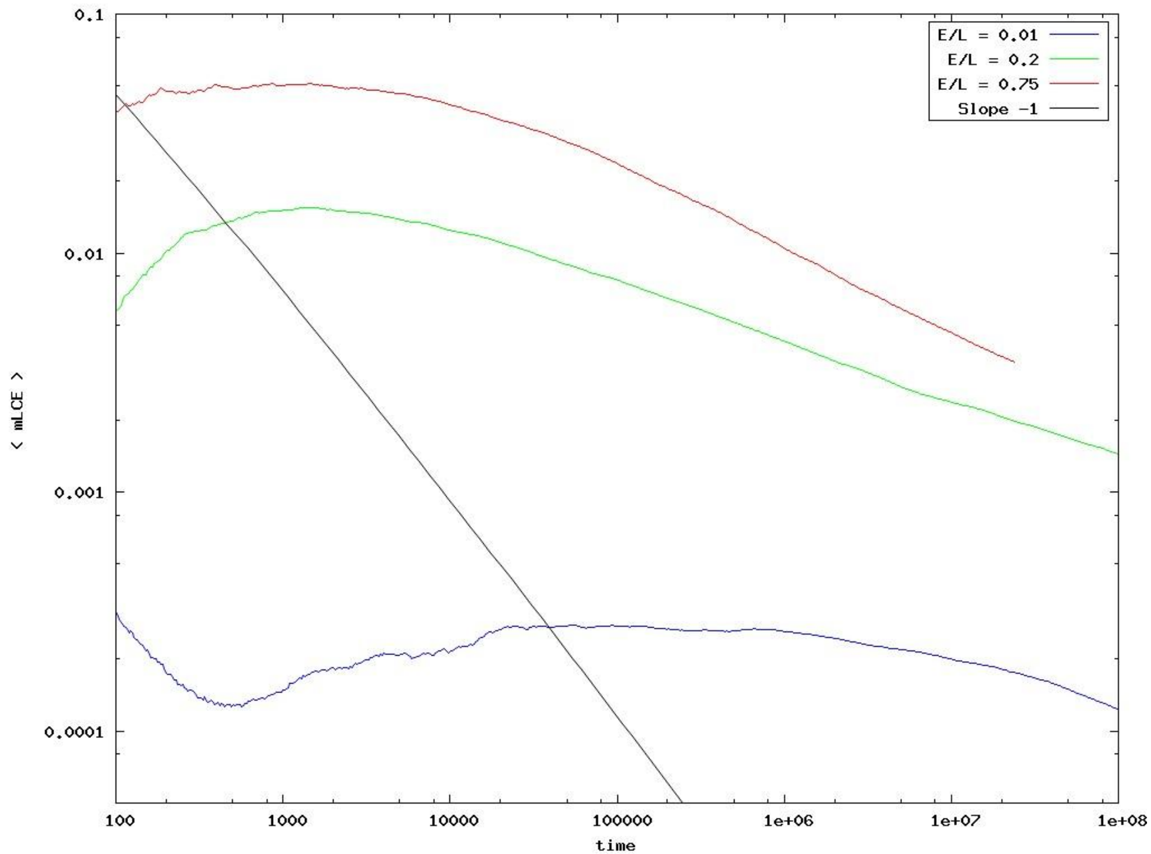


Block excitations:
Different spreading regimes

W = 4, E/L = 0.01, 0.2, 0.75

Average over 20 realizations





High order three part split symplectic integrators for the DNLS model

Ch.S., E. Gerlach, J. Bodyfelt, G. Papamikos & S. Eggl, 2012 (in preparation)

Three part split symplectic integrators for the DNLS model

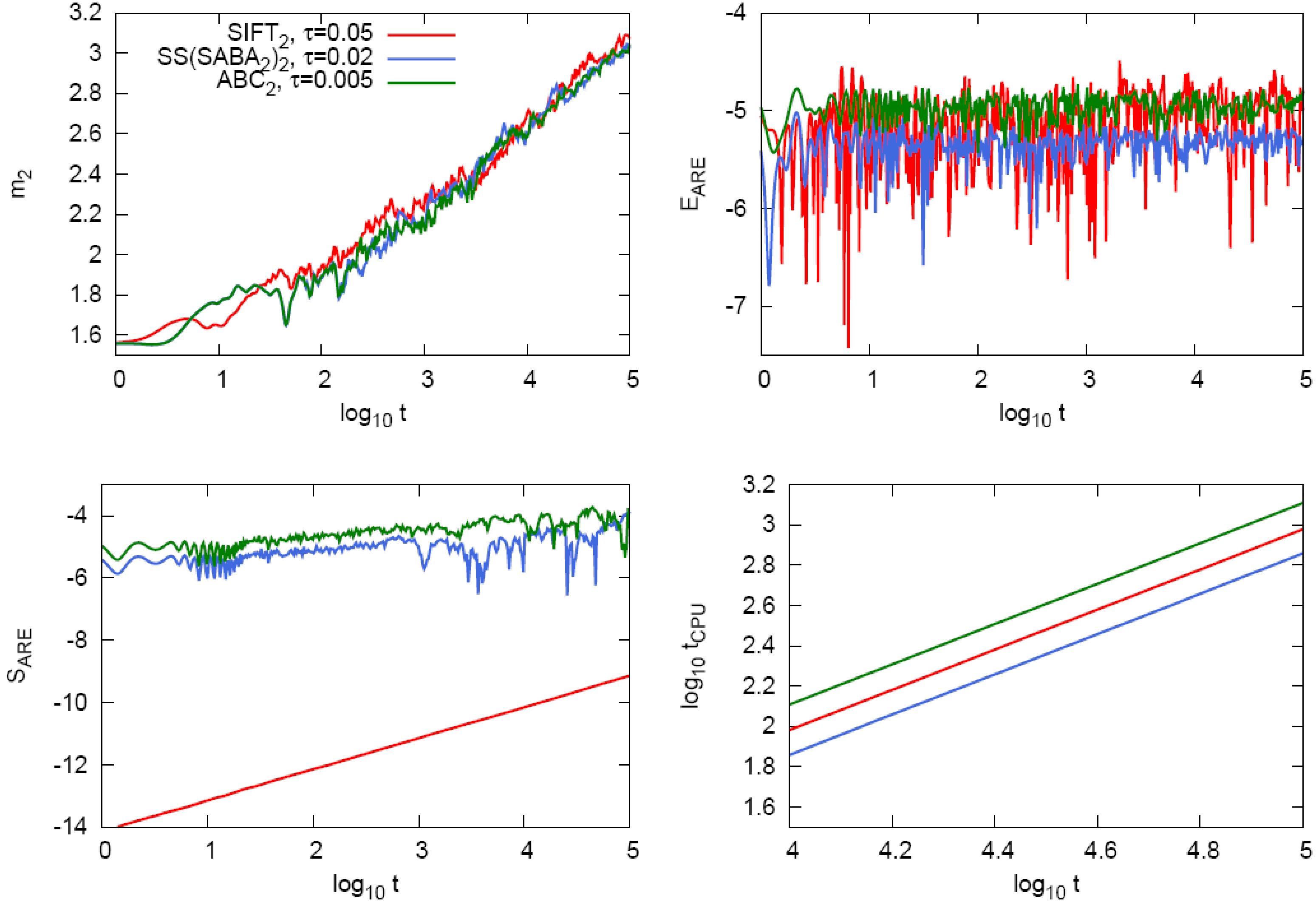
Three part split symplectic integrator of order 2, with 5 steps: ABC_2

$$H_D = \sum_l \Big(\underbrace{\frac{\varepsilon_l}{2}\left(q_l^2 + p_l^2\right) + \frac{\beta}{8}\left(q_l^2 + p_l^2\right)^2}_{A} \underbrace{- q_n q_{n+1}}_{B} \underbrace{- p_n p_{n+1}}_{C} \Big)$$

$$ABC_2 = e^{\frac{\tau}{2}L_A}\, e^{\frac{\tau}{2}L_B}\, e^{\tau L_C}\, e^{\frac{\tau}{2}L_B}\, e^{\frac{\tau}{2}L_A}$$

This low order integrator has already been used by e.g. Chambers, MNRAS (1999) – Goździewski et al., MNRAS (2008).

2^{nd} order integrators: Numerical results



4^{th} order symplectic integrators

Starting from any 2^{nd} order symplectic integrator S_{2nd}, we can construct a 4^{th} order integrator S_{4th} using a composition method [Yoshida, Phys. Let. A (1990)]:

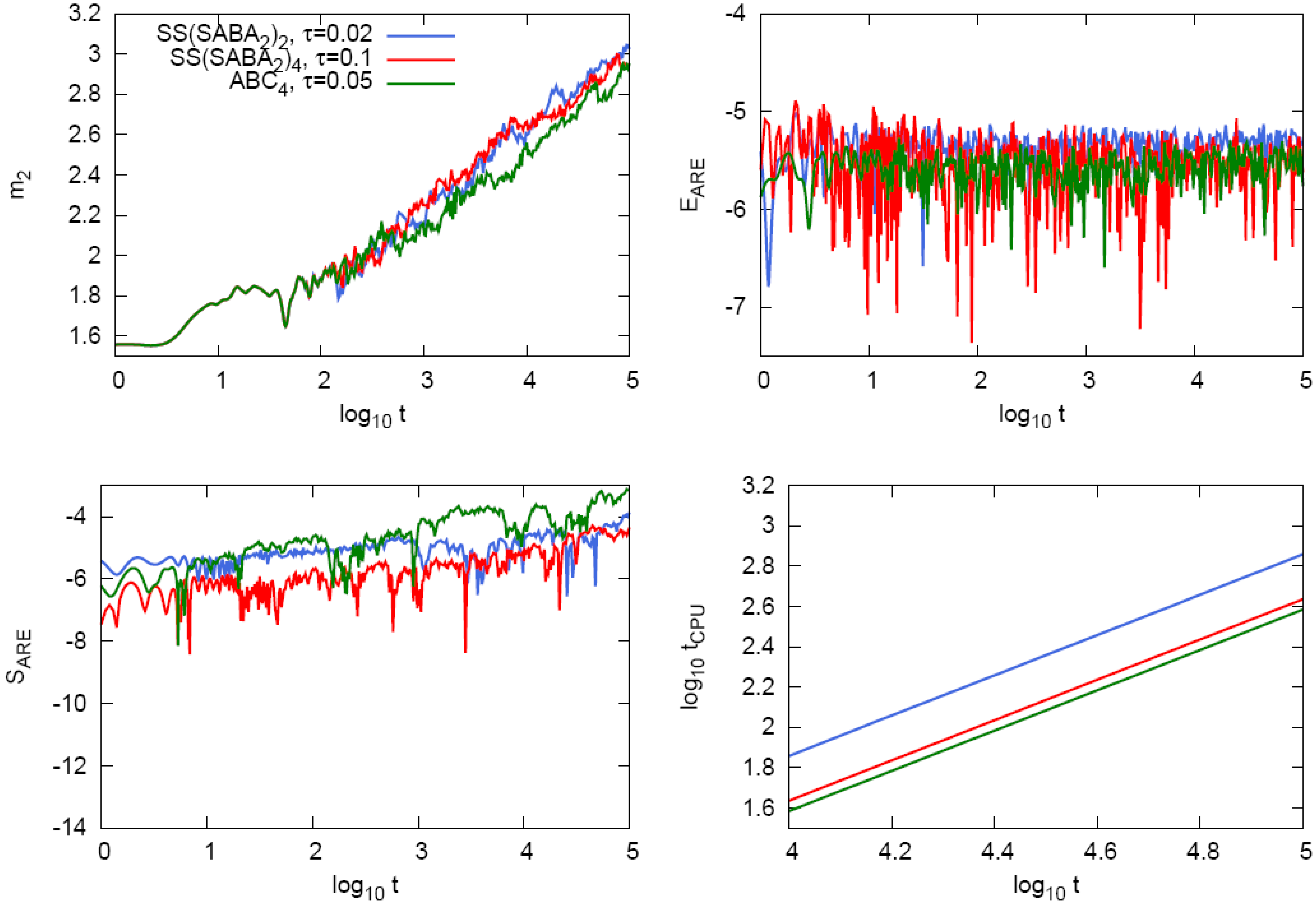
$$S_{4th}(\tau) = S_{2nd}(x_1\tau) \times S_{2nd}(x_0\tau) \times S_{2nd}(x_1\tau)$$

$$x_0 = -\frac{2^{1/3}}{2-2^{1/3}}, \qquad x_1 = \frac{1}{2-2^{1/3}}$$

Starting with the 2^{nd} order integrators $SS(SABA_2)_2$ and ABC_2 we construct the 4^{th} order integrators:

- **$SS(SABA_2)_4$ with 37 steps**
- **ABC_4 with 13 steps**

4th order integrators: Numerical results



Outlook

- **Disordered nonlinear lattices:**
  - ✓ **Quantify the chaotic behavior of energy spreading: Find theoretical or empirical laws for the evolution of Lyapunov exponents.**
  - ✓ **Use covariant Lyapunov vectors and frequency map analysis in order to study the chaotic nature of energy spreading: Do 'chaotic hot spots' exist?**
  - ✓ **Determine the limiting states of wave packets.**
  - ✓ **Identification of the selftrapped and spreading parts of wave packets.**
  - ✓ **Extension to higher spatial dimensions, and interactions beyond nearest neighbors.**

- **Three part split symplectic integrators**
  - ✓ **Computation of chaos indicators for the DNLS model.**
  - ✓ **Different techniques for constructing high order integrators.**
  - ✓ **Investigate the possible use of corrector terms.**
  - ✓ **Applications to other dynamical systems (e.g. models studied at UCT).**

- **Chaos detection techniques**
  - ✓ **Behavior of $GALI_k$ indices for time dependent Hamiltonians, dissipative systems, and time series.**
  - ✓ **Computation of the spectrum of LCEs using the compound matrix theory.**
  - ✓ **Review paper: Comparative study of the various existing methods.**

Main references

- Disordered systems

  - ✓ S. Flach, D.O. Krimer, Ch.S. (2009) PRL, 102, 024101
  - ✓ Ch.S., D.O. Krimer, S. Komineas, S. Flach (2009) PRE, 79, 056211
  - ✓ Ch.S., S. Flach (2010) PRE, 82, 016208
  - ✓ T.V. Laptyeva, J.D. Bodyfelt, D.O. Krimer, Ch.S., S. Flach (2010) EPL, 91, 30001
  - ✓ J.D. Bodyfelt, T.V. Laptyeva, Ch.S., D.O. Krimer, S. Flach (2011) PRE, 84, 016205
  - ✓ J.D. Bodyfelt, T.V. Laptyeva, G. Gligoric, Ch.S., D.O. Krimer, S. Flach (2011) Int. J. Bifurc. Chaos, 21(8), 2107
  - ✓ Ch.S., I. Golias, S. Flach (2012) in preparation

- TM method

  - ✓ Ch.S. & Gerlach E. (2010) Phys. Rev. E, 82, 036704
  - ✓ Gerlach E. & Ch.S. (2011) Discr. Cont. Dyn. Sys. Supp., 475
  - ✓ Gerlach E., Eggl S. & Ch.S. (2012) Int. J. Bifurc. Chaos, 22(9), 1250216

- Chaos indicators: Lyapunov exponents, GALI

  - ✓ Ch.S., Bountis T. C. & Antonopoulos Ch. (2007) Physica D, 231, 30-54
  - ✓ Ch.S., Bountis T. C. & Antonopoulos Ch. (2008) Eur. Phys. J. Sp. Top., 165, 5-14
  - ✓ Ch.S. (2010) Lect. Notes Phys., 790, 63
  - ✓ Manos T., Ch.S. & Antonopoulos Ch. (2012) Int. J. Bifurc. Chaos, 22(9), 1250218
  - ✓ Manos T., Bountis T. & Ch.S. (2012) J. Phys. A, in press

- Three part split symplectic integrators

  - ✓ Ch.S, Gerlach E., Bodyfelt J., Papamikos G. & Eggl S. (2012) in preparation

Summary

- **Multidimensional disordered nonlinear lattices :**
  - ✓ **The use of symplectic schemes allow us to follow their evolution for very long time intervals.**
  - ✓ **We predicted theoretically and verified numerically the existence of different dynamical behaviors: a) Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$, b) Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2} \longrightarrow m_2 \sim t^{1/3}$, c) Selftrapping Regime: $\delta>\Delta$**
  - ✓ **Generality of results: a) Two different models: KG and DNLS, b) Predictions made for DNLS are verified for both models.**

- **Numerical schemes based on symplectic integrators can be used for the efficient integration of the variational equations of multidimensional Hamiltonian systems.**
  - ✓ **Our results suggest that Anderson localization is eventually destroyed by the slightest amount of nonlinearity, since spreading does not show any sign of slowing down.**
  - ✓ **Energy spreading is a chaotic phenomenon, as Lyapunov exponent estimators decrease following an evolution different than in the case of regular motion.**

- **Three part split symplectic integrators**
  - ✓ **Proved to be efficient integration methods suitable for the integration of the DNLS model.**
  - ✓ **The 4^{th} order integrator allows integration of the DNLS model for very long times.**
  - ✓ **A systematic way of constructing high order three part split symplectic integrators was introduced.**